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Algebroid plane curves whose Milnor and Tjurina numbers differ by one or two

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Abstract. In this paper we describe all irreducible plane algebroid curves, defined over an algebraically closed field of characteristic zero, modulo analytic equivalence, having the property that the difference between their Milnor and Tjurina numbers is 1 or 2. Our work extends a previous result of O. Zariski who described such curves when this difference is zero.

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1. Introduction

In this section we begin with some definitions so that we can introduce our results. Let K be an algebraically closed field of characteristic zero, and let C be an algebroid plane curve defined by an irreducible power series f such that f and its partial derivatives f_X and f_Y are in the maximal ideal of $K[[X,Y]]$. The local ring of C is

$$\mathcal{O} = \mathcal{O}_f = \frac{K[[X,Y]]}{(f)} = K[[x,y]],$$

and the Milnor and Tjurina numbers of C are, respectively,

$$\mu(C) = \dim_K \frac{K[[X,Y]]}{(f_X, f_Y)} \quad \text{and} \quad \tau(C) = \dim_K \frac{K[[X,Y]]}{(f, f_X, f_Y)}.$$

We define the nonnegative integer

$$r(C) = \mu(C) - \tau(C).$$

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Two algebroid curves defined by f and g will be called *analytically equivalent* if there exists a K-algebra isomorphism $\mathcal{O}_f \simeq \mathcal{O}_g$.

To motivate our results, recall the following theorem due to Zariski [**7**]: *Let C be an irreducible algebroid plane curve. Then $r(C) = 0$, or $\mu(C) = \tau(C)$, if and only if C is analytically equivalent to the curve $Y^n - X^m$, for two coprime integers n and m greater than* 1.

In particular, Zariski's theorem suggests that the smaller $r(C)$ the more special the curve. The aim of this paper is to characterize, up to analytic equivalence, the curves C for which $r(C) = 1$ or $r(C) = 2$. Our characterization is given in Theorems 7, 12 and 17, below. Our method involves a refined analysis of the module of differentials of the local ring of the curves.

In Section 2 we give some definitions and recall some results. We also establish, in Proposition 1, a lower bound on $r(C)$ in terms of some integers associated to C. Finally, we sketch a proof of a result, Proposition 2, due to Azevedo [**2**], but unpublished; it is an essential ingredient in the proof of the converse in Theorem 17. In Section 3 we prove, in Corollary 6, that, if $r(C) \leq 2$, then $g \leq 2$ where g is the genus of C as defined in Section 2. Thus we have a severe constraint on the semigroup of C when its Milnor and Tjurina numbers differ by 1 or 2.

In Section 4 we describe all irreducible algebroid plane curves C with $r(C) = 1$. We show in Theorem 7 that $r(C) = 1$ only when $g = 1$, and that, up to analytic equivalence, there is just one class of such C with given semigroup. In Section 5 we treat the case $r(C) = 2$ and $g = 2$. The analysis is quite simple, and in Theorem 12 we show that this case occurs only for C with very special semigroups, and again that, for each semigroup, there is only one analytic class of C. In Section 6 we treat the more involved case $r(C) = 2$ and $g = 1$. Finally, in Theorem 17, we describe all analytic classes of these C.

2. Semigroups, parametrizations and differentials

Let $\mathcal{O}$ be the local ring of an irreducible algebroid plane curve C. Let $\tilde{\mathcal{O}}$ be the integral closure of $\mathcal{O}$, and consider its discrete normalized valuation ν. The value semigroup (or shortly the semigroup) of C is the semigroup of the natural numbers, $\mathbb{N}$, given by

$$S = \nu(\mathcal{O}).$$

Let $c = 2l(\frac{\tilde{\mathcal{O}}}{\mathcal{O}})$, where $l(M)$ means the length of a module M, be the conductor of C. Since C is assumed to be irreducible, it is well known that c is equal to the Milnor number $\mu(C)$ (see [**6**, Thm. 1]). It is also characterized by the following arithmetical property (see [8, Prop. 1.2]):

$$c-1 \notin S \quad \text{and} \quad c+n \in S, \qquad \forall n \geq 0.$$

The semigroup S is determined by its complement in $\mathbb{N}$. The set $\mathbb{N} \setminus S$ is a subset of $[0, c-1]$, hence finite and its elements are called the *gaps* of S.

We denote by $\mathcal{O}d\mathcal{O}$ the module of differentials of $\mathcal{O}$, that is, the $\mathcal{O}$-module generated by dx and dy, modulo the relation $f_X dx + f_Y dy = 0$. It is well known (see [7]) that

$$r(C) = \dim_K \left(\frac{\mathcal{O}d\mathcal{O}}{d\mathcal{O}} \right) = \#\big(\nu(\mathcal{O}d\mathcal{O}) \setminus \nu(d\mathcal{O})\big),$$

where ν is the obvious extension of the valuation ν of $\tilde{\mathcal{O}}$ to $\tilde{\mathcal{O}}d\tilde{\mathcal{O}}$. Therefore, the integer $r(C)$ can be interpreted as the maximum number of linearly independent nonexact differentials, modulo exact differentials.

Observe for future use that any element in $\nu(\mathcal{O}d\mathcal{O}) \setminus \nu(d\mathcal{O})$ plus 1 is a gap of S.

Since $\tilde{\mathcal{O}} \simeq K[[t]]$, we may represent C parametrically as follows:

$$x = t^n, \quad y = t^m + a_{m+1}t^{m+1} + \cdots,$$

where we may assume $n < m$, m is not a multiple of n, and the exponents n, m and the j such that $a_j \neq 0$ have no common nontrivial divisors. The valuation ν computes the orders of power series with respect to the parameter t.

Zariski's curve $X^m - Y^n$ corresponds to the monomial curve

$$x = t^n, \quad y = t^m.$$

There are two sequences (e_i) and (β_i) of integers, associated to an algebroid plane curve C, defined in terms of a parametrization as follows:

$$\begin{aligned} e_0 &= \beta_0 = n, \\ \beta_i &= \min\{j;\, j \not\equiv 0 \bmod e_{i-1} \ \text{ and } \ a_j \neq 0\}, \\ e_i &= \gcd\{e_{i-1}, \beta_i\}. \end{aligned}$$

It follows that $\beta_1 = m$. Since the relevant exponents in a parametrization of C are coprime, there exists an integer g, called the *genus* of C, such that $e_{g-1} \neq 1$ and $e_g = 1$. The integers $\beta_0, \beta_1, \ldots, \beta_g$ are called the *characteristic exponents* of C.

For example, the curve $Y^n - X^m$ in Zariski's theorem is of genus 1.

Let us define integers n_i as follows: $n_0 = 1$ and for $i = 1, \dots, g$,

$$e_{i-1} = n_i e_i.$$

It follows from this definition that $n = n_1 \cdots n_g$.

Zariski has shown in [**8**, Theorem 3.9] that the semigroup S of the curve C, represented parametrically as above, is minimally generated by the set of integers $\{v_0, v_1, \dots, v_g\}$, defined inductively by

$$v_i = n_{i-1}v_{i-1} + \beta_i - \beta_{i-1} \text{ for } i = 1, \dots, g$$

where v_0 is the multiplicity n of C. For this reason the integer g is also called the genus of the semigroup S.

It follows easily from the above formulas that $v_1 = m$ (the same m in the above parametrization), and this is the smallest element in S not divisible by n. It also follows that

$$e_i = \gcd\{e_{i-1}, v_i\} \text{ for } i = 1, \dots, g.$$

Since the β_i's may be determined by the v_i's through the above formulas, it follows that the characteristic integers do not depend upon the particular parametrization we have chosen for C.

It is well known (see [**1**, Lemma I.2.4]) that any integer t may be written in a unique way as

$$t = t_1 v_1 + \cdots + t_g v_g - t_0 v_0, \tag{1}$$

where $t_0, \dots, t_g$ are integers such that $0 \le t_i \le n_i - 1$ for $i = 1, \dots, g$. So, with this representation, we have that $t \in S$ if and only if $t_0 \le 0$.

From the above relations among the integers v_i and β_j we get that

$$v_i > n_{i-1}v_{i-1} \text{ for } i = 1, \dots, g. \tag{2}$$

Since $c - 1$ is the biggest gap in S, it follows easily from (1) that

$$c = (n_g - 1)v_g + \cdots + (n_1 - 1)v_1 - v_0 + 1. \tag{3}$$

If $r(C) \neq 0$, it is shown in [**7**] that the analytic invariant,

$$\lambda = \min\left(\nu(\mathcal{O}d\mathcal{O}) \setminus \nu(d\mathcal{O})\right) - n + 1,$$

is such that

$$\lambda, \lambda + n \notin S \text{ and } v_1 < \lambda \le \beta_2 = v_2 - v_1(n_1 - 1). \tag{4}$$

Furthermore, C is analytically equivalent to a curve with a parametrization of the form

$$x = t^n, \quad y = t^m + t^\lambda + \cdots .$$

From now on, we will refer to λ as *Zariski's invariant*.

We will assume from now on that $r(C) \neq 0$, and our curve C has a parametrization as above. For these curves, a nonexact differential with minimal value $\lambda + n - 1$ is given by

$$\omega = mydx - nxdy. \tag{5}$$

Below we will always write

$$\lambda = \lambda_1 v_1 + \cdots + \lambda_g v_g - \lambda_0 v_0, \tag{6}$$

where $0 \leq \lambda_i \leq n_i - 1$, for $i = 1, \ldots, g$. We have that $\lambda_0 \geq 2$, since $\lambda \notin S$ and $\lambda + n \notin S$.

Proposition 1. *Suppose that $r(C) \neq 0$. With the above notation we have*

$$r(C) \geq (\lambda_0 - 1)(n_1 - \lambda_1) \cdots (n_g - \lambda_g).$$

Proof. Let $z_i \in \mathcal{O}$ such that $\nu(z_i) = v_i$, $i = 1, \ldots, g$. With ω as in (5), define

$$\omega_\alpha = x^{\alpha_0} z_1^{\alpha_1} \ldots z_g^{\alpha_g} \omega,$$

where $\alpha = (\alpha_0, \ldots, \alpha_g) \in \mathbb{N}^{g+1}$. We then have

$$\nu(\omega_\alpha) = \alpha_0 n + \alpha_1 v_1 + \cdots + \alpha_g v_g + n + \lambda - 1;$$

that is,

$$1 + \nu(\omega_\alpha) = (\alpha_0 + 1 - \lambda_0)n + (\alpha_1 + \lambda_1)v_1 + \cdots + (\alpha_g + \lambda_g)v_g.$$

If we choose $\alpha = (\alpha_0, \ldots, \alpha_g)$ such that $\alpha_0 + 1 - \lambda_0 < 0$, and $0 \leq \alpha_i + \lambda_i \leq n_i - 1$, $i = 1, \ldots, g$, then $1 + \nu(\omega_\alpha) \notin S$, so ω_α will be a nonexact differential. For this, it is enough to take $0 \leq \alpha_0 \leq \lambda_0 - 2$ and $0 \leq \alpha_i \leq n_i - \lambda_i - 1$ for $i = 1, \ldots, g$.

Since distinct $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_g)$, satisfying the above conditions give distinct values for the differentials ω_α, it follows that there are at least

$$(\lambda_0 - 1)(n_1 - \lambda_1) \cdots (n_g - \lambda_g)$$

nonexact differentials, linearly independent over K modulo exact differentials. The result is then established. □

The next result will be useful below, and since we don't have any accessible reference for it, we include a sketch of its proof.

Proposition 2. [Azevedo [**2**]] *We have the following equality of K-vector spaces:*

$$\mathcal{O}d\mathcal{O} = \mathcal{O}\omega + d\mathcal{O},$$

where ω is as in (5).

Proof. It is not difficult to verify that given two series $A, B \in K[[X, Y]]$, and two positive integers r and s, there exist $G, C \in K[[X, Y]]$ such that

$$G_X = A + rXC, \qquad G_Y = B - sYC$$

(to determine G and C just integrate the first equation with respect to X and then substitute the series G so found in the second equation).

Let $\eta \in \mathcal{O}d\mathcal{O}$. Then $\eta = A(x, y)dx + B(x, y)dy$, where $A(X, Y)$, $B(X, Y)$ are in $K[[X, Y]]$. If we put $r = n$ and $s = m$, and if $G(X, Y)$ and $C(X, Y)$ are as above, then we have

$$\begin{aligned} dG(x, y) &= G_X(x, y)dx + G_Y(x, y)dy \\ &= A(x, y)dx + B(x, y)dy + (nxdy - mydy)C(x, y) \\ &= \eta - \omega C(x, y). \end{aligned}$$

Hence,

$$\eta = C(x, y)\omega + dG(x, y) \in \mathcal{O}\omega + d\mathcal{O}. \qquad \square$$

Proposition 2 immediately gives the next result (see also [**8**, Ch. V, Lemme 4.2]).

Corollary 3. *If $\eta \in \mathcal{O}d\mathcal{O} \setminus d\mathcal{O}$ and $\nu(\eta) > \nu(\omega)$, then $\nu(\eta) \geq \nu(\omega) + n$, with strict inequality if $\nu(x\omega) \in \nu(d\mathcal{O})$.*

We now state some criteria for eliminating parameters due to Ebey [**3**] and to Zariski [**8**] (see [**8**, Ch. III, Prop. 1.2; Ch. IV, Lemme 2.6 and Prop. 3.1]).

If $a_s t^s$, with $s > \lambda$ and $a_s \neq 0$, is a term of y in the parametrization of C, and if one of the following conditions holds,

(EC1) $s \in S$, or

(EC2) $s + n = lm$, for some $l \in \mathbb{N}$, or

(EC3) $s - \lambda$ is in the semigroup generated by n and m,

then C is analytically equivalent to a curve with a parametrization of the same form, but with $a_s = 0$ and a_i unchanged for $i < s$.

3. Constraints on the semigroup for low $r(C)$

In this section we show that if $r(C)$ is small, then C must have a small genus g.

Suppose that $r(C) \neq 0$ and let λ be Zariski's invariant of C written as in (6). Define the integer j as follows

$$j = \max\{i; \lambda_i \neq 0\}.$$

Since $\lambda_0 \geq 2$ and $\lambda > 0$, we certainly have $j \geq 1$ and $\lambda_j \geq 1$. It follows that

$$\lambda \geq \lambda_1 v_1 + \cdots + \lambda_{j-1} v_{j-1} + v_j - \lambda_0 v_0, \tag{7}$$

and from Proposition 1 that

$$r(C) \geq (\lambda_0 - 1) n_{j+1} \cdots n_g. \tag{8}$$

Proposition 4. *With notation as above, we have* $r(C) \geq \dfrac{n}{n_j}$.

Proof. **Case** $j = 1$. From (8) we have

$$r(C) \geq n_2 \cdots n_g = \frac{n}{n_1}.$$

Case $j = 2$. a) Suppose $\lambda_1 \geq 1$. From (4) and (7) we have

$$v_2 - (n_1 - 1) v_1 \geq \lambda \geq v_2 + v_1 - \lambda_0 v_0.$$

If we write $\dfrac{m}{n} = \dfrac{m_1}{n_1}$ with $\gcd(m_1, n_1) = 1$, then the above inequality yields

$$\lambda_0 v_0 \geq n_1 v_1 = m_1 v_0.$$

Hence $\lambda_0 \geq m_1 > n_1$, and therefore $\lambda_0 - 1 \geq n_1$. From this and (8) we get

$$r(C) \geq n_1 n_3 \cdots n_g = \frac{n}{n_2}.$$

b) Suppose $\lambda_1 = 0$. From Proposition 1, we have

$$r(C) \geq n_1 (n_2 - \lambda_2) n_3 \cdots n_g \geq n_1 n_3 \cdots n_g = \frac{n}{n_2}.$$

Case $j \geq 3$. From (4) and (7) we have

$$v_2 - (n_1 - 1) v_1 \geq \lambda \geq v_j - \lambda_0 v_0.$$

Hence, in view of (2),

$$\begin{aligned}\lambda_0 v_0 &\geq v_j - v_2 + (n_1 - 1)v_1 > n_{j-1}v_{j-1} - v_2 + v_1 \\ &= (n_{j-1} - 1)v_{j-1} + (v_{j-1} - v_2) + v_1 > n_{j-2} \cdots n_1 v_1 + v_1 \\ &> n_{j-2} \cdots n_1 v_0 + v_0.\end{aligned}$$

Hence $\lambda_0 - 1 > n_{j-2} \cdots n_1$. From Proposition 1, we get

$$\begin{aligned}r(C) &> n_{j-2} \cdots n_1 n_{j-1}(n_j - \lambda_j)n_{j+1} \cdots n_g \\ &\geq n_1 \cdots n_{j-2} n_{j-1} n_{j+1} \cdots n_g = \frac{n}{n_j}.\end{aligned}$$ □

Corollary 5. *If $r(C) \neq 0$, then $r(C) \geq 2^{g-1}$.*

Proof. The assertion follows immediately from Proposition 4 because $n_i \geq 2$ for all $i = 1, \ldots, g$ and $n = n_1 \cdots n_g$. □

Corollary 6. *If $r(C) \leq 3$, then $g \leq 2$.*

Proof. If $r(C) = 0$, then $g = 1$ by Zariski's result stated in the introduction. If $1 \leq r(C) \leq 3$, then Corollary 5 implies $2^{g-1} \leq 3$; whence, $g \leq 2$. □

4. Singularities with $r(C) = 1$

If $r(C) = 1$, then Zariski's invariant λ is well defined, and Proposition 1 gives

$$1 = r(C) \geq (\lambda_0 - 1)(n_1 - \lambda_1). \tag{9}$$

Theorem 7. *Let C be an algebroid irreducible plane curve with semigroup of values S, and Zariski's invariant λ. We have that $r(C) = 1$ if and only S is generated by two coprime integers n and m with $n < m$, and $\lambda = (n-1)m - 2n$. In this case, C is analytically equivalent to the curve given parametrically by*

$$x = t^n, \quad y = t^m + t^{(n-1)m-2n}.$$

Proof. Assume that $r(C) = 1$. From Corollary 5, it follows that $g = 1$. If we denote by n and m, with $n < m$, the generators of S, we get from (9) that $\lambda_0 = 2$ and $\lambda_1 = n - 1$. So

$$\lambda = (n-1)m - 2n.$$

Conversely, if S is generated by n and m, and if $\lambda = (n-1)m - 2n$, then

$$\lambda + n = (n-1)m - n = c - 1,$$

where c is the conductor of the semigroup S, which shows that $\lambda + n$ is the biggest gap of S. This implies that the differential ω in (5) has the highest possible value. Since ω has the least value among the nonexact differentials, it follows that there is no room for other linearly independent nonexact differentials modulo exact differentials. This proves that $r(C) = 1$.

In view of (**EC1**) and (**EC2**), we have just proved that $r(C) = 1$ if and only if C has a parametrization of the form

$$x = t^n, \quad y = t^m + bt^{(n-1)m-2n},$$

where $b \in K$ is nonzero. Changing variables via $\tau = \zeta t$, $x' = \zeta^n x$ and $y' = \zeta^m y$ with $\zeta^{(n-2)m-2n} = b$, we see we can take $b = 1$ in the above parametrization. □

Our result shows that there is a severe constraint on the semigroup of an algebroid plane curve with $r(C) = 1$: it must have genus one. Furthermore, for every such semigroup there is one and only one class, modulo analytic equivalence, of algebroid plane curves with $r(C) = 1$.

Corollary 8. *Let C be an algebroid irreducible plane curve. Then $r(C) = 1$ if and only if C is analytically equivalent to a curve determined by $Y^n - X^m + X^{m-2}Y^{n-2}$, where n and m are coprime integers greater or equal than two.*

Proof. It is easy to show that the semigroup of the curve defined by the above polynomial is generated by n and m. In view of the unicity statement contained in Theorem 7, we have only to verify that this curve has $\lambda = (n-1)m - 2n$. This follows from [**5**, Theorem 1.5, where we put $s = \lambda - m$]. □

5. Singularities with $r(C) = 2$ and $g = 2$

We shall assume in this section that $r(C) = 2$ and $g = 2$. The case $g = 1$ will be analyzed separately in the next section.

From Proposition 1, we have

$$0 < (\lambda_0 - 1)(n_1 - \lambda_1)(n_2 - \lambda_2) \le r(C) = 2. \tag{10}$$

Lemma 9. *If $g = 2$ and $r(C) = 2$, then $(\lambda_0 - 1)(n_1 - \lambda_1)(n_2 - \lambda_2) = 2$.*

Proof. Supppose that $(\lambda_0 - 1)(n_1 - \lambda_1)(n_2 - \lambda_2) = 1$. Then one should have $\lambda_0 = 2$, $\lambda_1 = n_1 - 1$ and $\lambda_2 = n_2 - 1$. So from (4) we get that

$$(n_2 - 1)v_2 + (n_1 - 1)v_1 - 2v_0 = \lambda \leq v_2 - v_1(n_1 - 1),$$

which yields the contradiction:

$$(n_2 - 1)v_2 + 2(n_1 - 1)v_1 \leq 2v_0,$$

since $n_1 - 1 > 0$, $n_2 - 1 > 0$, $v_1, v_2 > v_0$. □

Note that in the present case we have from Proposition 4 that

$$\frac{n_1 n_2}{n_j} = \frac{n}{n_j} \leq 2. \tag{11}$$

We have the following result

Lemma 10. *If $g = 2$ and $r(C) = 2$, then $j = 2$.*

Proof. Suppose that $j = 1$. From the definition of j we must have $\lambda_2 = 0$, and from (11) we have that $n_2 \leq 2$, and since $n_2 \geq 2$, it follows that $n_2 = 2$.

Since $e_1 = n_2 e_2$ and $e_2 = 1$, it follows that $e_1 = 2$ and therefore our semigroup S is of the form $S = \langle 2p, 2q, v_2 \rangle$, with $p < q$, p and q coprime, $n_1 = p$ and $v_2 > n_1 v_1 = 2pq$. It follows, for some positive and odd integer d, that

$$S = \langle 2p, 2q, 2pq + d \rangle.$$

The algebroid irreducible plane curves with such a semigroup S have been studied by Luengo and Pfister in **[4]**, where they prove that any such curve has $\tau(C) = c - (p - 1)(q - 1)$. Since in our case $\tau(C) = c - 2$, it follows that $p = 2$ and $q = 3$, and therefore $S = \langle 4, 6, 12 + d \rangle$.

Since $n_2 = 2$ and $\lambda_2 = 0$, from inequality (10) it follows that $\lambda_0 = 2$ and $\lambda_1 = n_1 - 1 = 1$. Hence we obtain the contradiction,

$$0 < \lambda = \lambda_1 v_1 - \lambda_0 v_0 = 1 \times 6 - 2 \times 4 < 0.$$ □

From now on, in this section, we will assume $j = 2$, and therefore from (11) we must have $n_1 = 2$.

Proposition 11. *Let $g = 2$ and $r(C) = 2$, then $S = \langle 4, 6, v_2 \rangle$, with v_2 odd, $v_2 \geq 13$, and $\lambda = \beta_2 = v_2 - 6$.*

Proof. From Lemma 9 and from the observation after Lemma 10, we know that $(\lambda_0 - 1)(n_1 - \lambda_1)(n_2 - \lambda_2) = 2$, and $n_1 = 2$. So it remains to analyze the few cases below.

Case A) $\lambda_0 = 3$, $\lambda_1 = n_1 - 1 = 1$ and $\lambda_2 = n_2 - 1$. From (4) we have

$$(n_2 - 1)v_2 + v_1 - 3v_0 \leq v_2 - v_1.$$

From this inequality and from (2), for $i = 2$, we get that $(2n_2 - 2)v_1 \leq 3v_0$.

This implies $n_2 = 2$ $(= e_1)$, so $v_0 = n = n_1 n_2 = 4$, and v_1 is even. In this case the above inequality also gives $2v_1 \leq 12$, hence $v_1 = 6$.

Since $v_2 > n_1 v_1$, it follows that $v_2 \geq 13$. On the other hand,

$$\lambda = \lambda_2 v_2 + \lambda_1 v_1 - \lambda_0 v_0 = v_2 + v_1 - 3v_0 = v_2 - 6 = \beta_2.$$

Case B) $\lambda_0 = 2$, $\lambda_1 = n_1 - 2 = 0$ and $\lambda_2 = n_2 - 1$. From (4) we have

$$(n_2 - 1)v_2 - 2v_0 \leq v_2 - v_1.$$

Therefore,

$$(n_2 - 2)v_2 + v_1 \leq 2v_0.$$

Hence $n_2 = 2$, and $v_1 \leq 2v_0 = 2n = 8$. Therefore $v_1 = 6$. On the other hand, if $\lambda < \beta_2$, then $e_1 = 2$ divides λ. Since $\lambda = v_2 - 2v_0$, it follows that $2|v_2$, a contradiction. Hence $\lambda = \beta_2$. Consequently,

$$\lambda = v_2 - 2v_0 = v_2 - v_1,$$

and therefore $6 = v_1 = 2v_0 = 8$, a contradiction. So this case doesn't occur.

Case C) $\lambda_0 = 2$, $\lambda_1 = n_1 - 1 = 1$ and $\lambda_2 = n_2 - 2$. This case also doesn't occur, because from (4) and (7) we get that

$$(n_2 - 2)v_2 + v_1 - 2v_0 \leq \lambda \leq v_2 - v_1.$$

Therefore,

$$(n_2 - 3)v_2 + 2v_1 \leq 2v_0.$$

Hence $n_2 = 2$. So we have $v_1 < \lambda = v_1 - 2v_0$, a contradiction. □

Theorem 12. *Let C be an irreducible plane algebroid curve C of genus g. Then $g = 2$ and $r(C) = 2$ if and only if C is analytically equivalent to a curve determined by the parametrization,*

$$x = t^4, \quad y = t^6 + t^\beta,$$

where $\beta > 6$ and β is odd.

Proof. This assertion follows from Proposition 11, and the classification of curves with semigroup $S = \langle 4, 6, v_2 \rangle$. The classification is found in [**8**, pp. 49–57]. □

Corollary 13. Let C be an irreducible plane algebroid curve C of genus g. Then $g = 2$ and $r(C) = 2$ if and only if C is analytically equivalent to a curve determined by $(X^2 + Y^3)^2 + XY^{(\beta+3)/2}$, for some odd integer $\beta > 6$.

Proof. This assertion follows from Theorem 12 and [**4**], where it is shown that $r(C) = 2$ for the curve $(X^2 + Y^3)^2 + XY^{(\beta+3)/2}$. □

6. Singularities with $r(C) = 2$ and $g = 1$

In this section we study the curves C such that $r(C) = 2$ and $g = 1$, thereby concluding the description of all irreducible algebroid plane curves with $r(C) = 2$.

Suppose that $r(C) = 2$ and $g = 1$. From Proposition 1 we get

$$2 = r(C) \geq (\lambda_0 - 1)(n - \lambda_1).$$

Hence there are three cases to consider.

Case A′) $\lambda_0 = 2$ and $\lambda_1 = n - 1$. In this case,

$$\lambda = (n-1)v_1 - 2v_0,$$

and from (3),

$$c = (n-1)v_1 - v_0 + 1.$$

Hence $c - \lambda = v_0 + 1 = n + 1$, so $c = \lambda + n - 1$. In this case, there is no room for a nonexact differential other than ω; therefore, this case doesn't occur.

Case B′) $\lambda_0 = 3$ and $\lambda_1 = n - 1$. In this case,

$$\lambda = (n-1)v_1 - 3v_0 = (n-1)m - 3n,$$

and from (3),

$$c = (n-1)v_1 - v_0 + 1 = (n-1)m - n + 1.$$

The only gaps of S at least $\nu(\omega) + 1$ are $(n-1)m - 2n$ and $(n-1)m - n$, and they correspond to the differentials ω and $x\omega$. Therefore, we have just proved that all the curves C with Zariski's invariant $\lambda = (n-1)m - 3n$ are such that $r(C) = 2$. Furthermore, each such curve is analytically equivalent to some member of the family

$$x = t^n, \quad y = t^m + t^\lambda + at^{\lambda+2n-m} + bt^{\lambda+n} + ct^{\lambda+2n},$$

with $a = 0$, if $m > 2n$. Now, using the criteria for eliminating parameters stated in Section 2, we get that C is analytically equivalent to the curve given by

$$x = t^n, \quad y = t^m + t^\lambda.$$

Case C′) $\lambda_0 - 1 = 1$ and $n - \lambda_1 = 2$. In this case,

$$\lambda = (n-2)v_1 - 2v_0 = (n-2)m - 2n, \tag{12}$$

and

$$c = (n-1)v_1 - v_0 + 1 = (n-1)m - n + 1.$$

Together, Equation (12) and the inequality $\lambda > m$ imply that $n \geq 4$ and that $m > 2n/(n-3)$. So the curve is given by a parametrization of the type

$$x = t^n, \quad y = t^m + t^\lambda + at^\mu + \cdots,$$

where μ and the higher exponents of t are gaps of S above λ, and are of the type $(n-2)m - n$, or $(n-1)m - jn$, $1 \leq j \leq [\frac{m}{n}] + 1$.

By (**EC2**) the term of order $(n-2)m - n$, in the above parametrization, may be eliminated, yielding the curve

$$x = t^n \quad y = t^m + t^{(n-2)m-2n} + a_1 t^{\mu_1} + a_2 t^{\mu_2} + \cdots, \tag{13}$$

where $\mu_i = (n-1)m - j_i n$ for $j_i \in \{1, \ldots, [\frac{m}{n}] + 1\}$. Now, we may assume that $j_1 \geq 3$, because if $j_1 \leq 2$, by (**EC3**), the curve (13) is reduced to the case $j_1 \geq 3$, or to the case $a_i = 0$ for all i, which will be studied later as a limit case.

Conversely, the curves determined by the parametrizations (13) have $r(C) \geq 2$, because ω and $y\omega$ are nonexact differentials with distinct values.

The following two lemmas will tell us which curves given in (13) must be excluded since they have $r(C) \geq 3$.

Lemma 14. *Let n and m be coprime integers such that $n \geq 5$ and $m > 2n/(n-3)$. Let C be a curve given by (13) with $j_1 \geq 4$ and $a_1 \neq 0$. Then $r(C) \geq 3$.*

Proof. We will show that in this case the differentials ω, $y\omega$ and

$$\omega' = mx\omega - n(m-\lambda)y^{n-3}dy,$$

are nonexact linearly independent over K, modulo exact differentials. To do so, it is sufficient to show that the value of ω' plus 1 is a gap distinct from $(n-2)m-n$ and $(n-1)m-n$.

A direct computation gives

$$\begin{aligned}\omega' = \big\{ & a_1mn(m-\mu_1)t^{2n+\mu_1-1} - n(m-\lambda)[\lambda+m(n-3)]t^{m(n-3)+\lambda-1}\\ & - n(m-\lambda)a_1[\mu_1+(n-3)]t^{m(n-3)+\mu_1-1}\\ & - n(m-\lambda)(n-3)[\lambda+m(n-4)/2]t^{m(n-4)+2\lambda-1}\\ & - n(m-\lambda)(n-3)[\mu_1+\lambda+m(n-4)]a_1t^{m(n-4)+\lambda+\mu_1-1}\\ & + mn(m-\mu_2)a_2t^{2n+\mu_2-1} - n(m-\lambda)[\mu_2+m(n-3)]a_2t^{m(n-3)+\mu_2-1}\\ & - n(m-\lambda)(n-3)[\mu_1+m(n-4)/2]a_1^2t^{m(n-4)+2\mu_1-1}\\ & - n(m-\lambda)(n-3)[\mu_2+\lambda+m(n-4)]t^{m(n-4)+\lambda+\mu_2-1} + \cdots\big\}dt,\end{aligned}$$

where the above terms are not necessarily in strictly increasing order.

Since $n \geq 5$ and $m > 2n/(n-3)$, we have

$$\nu(\omega') + 1 = \mu_1 + 2n = (n-1)m - (j_1-2)n,$$

a gap distinct from $(n-2)m-n$ and $(n-1)m-n$, proving the result. □

Lemma 15. *Let $m > 8$ be an integer coprime with $n = 4$. Let C be a curve given by* (13)*, with $a_1 \neq 0$. Then $r(C) \geq 3$ if either*

(i) $j_1 = 3$, *or* (ii) $j_1 \geq 5$, *or* (iii) $j_1 = 4$ *and* $a_1 \neq (3m-8)/2m$.

Proof. Here the differential ω' considered in the proof of Lemma 14 becomes

$$\begin{aligned}\omega' = & \big\{4a_1m(4j_1-2m)t^{3m+7-4j_1} + 4(m-8)(3m-8)t^{3m-9}\\ & + 16(m-8)(m-j_1)a_1t^{4(m-j_1)-1} + 8(m-8)(m-4)t^{4m-16} + \cdots\big\}dt.\end{aligned}$$

(i) Let $j_1 = 3$. Then,

$$\omega' = \{4(m-8)(3m-8)t^{3m-9} + 8a_1m(6-m)t^{3m-5} \\ + 16(m-8)(m-3)a_1t^{4m-13} + 8(m-8)(m-4)t^{4m-16} + \cdots\}dt.$$

In this case, since $m > 8$, we have $\nu(\omega') = 3m-9$. So $\nu(\omega')+1 = (n-1)m-2n$, which is a gap different from $\nu(\omega)+1 = (n-2)m-n$ and from $\nu(y\omega)+1 = (n-1)m-n$. Therefore $r(C) \geq 3$.

(ii) Let $j_1 \geq 5$. Then $\nu(\omega') = 3m+7-4j_1$; that is, $\nu(\omega')+1 = (n-1)m-(j_1-2)n$ is a gap different from $\nu(\omega)+1$ and from $\nu(y\omega)+1$. Therefore $r(C) \geq 3$.

(iii) Suppose $j_1 = 4$. In this case,

$$\omega' = \{4(m-8)[(3m-8)-2a_1m]t^{3m-9} + 16(m-8)(m-4)a_1t^{4m-17} \\ + 8(m-8)(m-4)t^{4m-16} + \cdots\}dt.$$

If moreover, $a_1 \neq (3m-8)/2m$, then $\nu(\omega') = 3m-9$ (recall that $m > 8$). Therefore, $\nu(\omega')+1 = (n-1)m-2n$ is a gap different $\nu(\omega)+1$ and from $\nu(y\omega)+1$. Hence, once again, $r(C) \geq 3$. □

So if the curve in (13) is such that $r(C) = 2$, then we must have either

(a) $n \geq 5$ and $\mu_1 = (n-1)m-3n$, or

(b) $n = 4$, $\mu_1 = (n-1)m-4n$, and $a_1 = (3m-8)/2m$.

In Case (a), by (**EC3**), the other terms t^{μ_i} for $i \geq 2$ may be eliminated, yielding a curve with a parametrization,

$$x = t^n, \quad y = t^m + t^{(n-2)m-2n} + at^{\mu}, \tag{14}$$

where $a \in K$.

In Case (b), by (**EC3**), the terms t^{μ_i} for $i \geq 3$ may be eliminated, yielding a curve with a parametrization,

$$x = t^4, \quad y = t^m + t^{2m-8} + \frac{3m-8}{2m}t^{\mu_1} + at^{\mu_2}, \tag{15}$$

where $\mu_2 = 3m-12$ and $a \in K$.

Conversely, we have the following result.

Proposition 16. *Let n and m be coprime integers such that $n \geq 4$ and such that $m > 2n/(n-3)$. Let C be a curve given either by*

(i) *Parametrization* (14), *with $n \geq 5$ and $\mu = (n-1)m - 3n$, or by*

(ii) *Parametrization* (15), *with $\mu_1 = 3m - 16$ and $\mu_2 = 3m - 12$.*

Then $r(C) = 2$.

Proof. In either case, it is sufficient to prove the result for $a \neq 0$; indeed, since $r(C)$ is semicontinuous, it will follow that $r(C) \leq 2$ if $a = 0$. Now, $r(C) \neq 0, 1$ because of Theorem 7 and Zariski's result stated in the introduction.

We have to show that there isn't any differential in $\mathcal{O}d\mathcal{O}$ whose value plus one is different from $\nu(y\omega) + 1$, and is a gap of S between $(n-2)m - n$ and $c - 1$. These gaps are of the form $l_i = (n-1)m - in$, where $1 \leq i \leq 1 + [m/n]$.

Suppose $\omega' \in \mathcal{O}d\mathcal{O}$ is such that $\nu(\omega') + 1 = l_i$. By Proposition 2 we may write

$$\omega' = g\omega + dh, \tag{16}$$

for some $g, h \in \mathcal{O}$. Now, in view of the representation of the curve C given in (14) or (15), the Cartesian equation of C is in Weierstrass form. So by the Weierstrass Division Theorem, we may write any element of $\mathcal{O}$ in the form,

$$A_0(x) + A_1(x)y + \cdots + A_{n-1}(x)y^{n-1},$$

where, because of the uniqueness of the representation of an integer in the form (1), two distinct monomials in this expression have distinct values.

If $x^\alpha y^\beta$ is not in the set $\{y, x, x^2, \ldots, x^q\}$ where $q = [m/n]$, then all terms of the form $x^\alpha y^\beta \omega$ will have values above c. Also because we are looking for differentials ω' such that $\nu(\omega') \neq \nu(y\omega)$, and the higher terms in $y\omega$ have values above c, it follows that modulo $d\mathcal{O}$ we may assume g is a linear combination, with coefficients in K, of $x, x^2, \ldots, x^q$.

Suppose now that (i) holds. Then by (16) we have

$$\omega' = (b_1 x + \cdots + b_q x^q)\omega + d\Big(\sum c_{\alpha,\beta} x^\alpha y^\beta\Big), \tag{17}$$

for some $b_1, \ldots b_q$, and $c_{\alpha,\beta}$ in K.

Let k be the least integer such that $b_k \neq 0$, and consider the expression

$$x^k\omega = n(n-\lambda)t^{(n-2)m+(k-1)n-1} + an(m-\mu)t^{(n-1)m+(k-2)n-1}.$$

Suppose that $k = 1$. Since $(n-2)m + (k-1)n \in S$, the only possible value of ω' plus one is the only gap left, that is, $(n-1)m - n$. But in this case, $\nu(\omega') = \nu(y\omega)$.

Suppose that $k \geq 2$. Since, $(n-2)m + (k-1)n < (n-1)m + (k-2)n$, and both sides are in S, we see that the value of ω' may only come from a higher order term of an expression $dx^\alpha y^\beta$, for which $\alpha = k-1$ and $\beta = n-2$. Since,

$$x^{k-1}y^{n-2} = t^{(n-2)m+(k-1)n} + (n-2)[t^{(n-3)\lambda+(k-1)n} + at^{(n-3)m+\mu+(k-1)n}] + \cdots,$$

it follows that all higher terms of $dx^{k-1}y^{n-2}$ have value greater than c.

Suppose now that (ii) holds. Let k be an integer such that $1 \leq k \leq q$. In this situation we have

$$x^k\omega = \Big\{4(8-m)t^{2m+4(k-1)-1} + \frac{4}{m}(3m-8)(8-m)t^{3m+4(k-3)-1} + 8a(6-m)t^{3m+4(k-2)-1}\Big\}dt.$$

If $k \geq 3$, then plainly $x^k\omega$ is an exact differential. So we may write (17) as

$$\omega' = (b_1x + b_2x^2)\omega + d\Big(\sum c_{\alpha,\beta}x^\alpha y^\beta\Big).$$

Therefore,

$$\omega' = \Big\{4b_1(8-m)t^{2m-1} + b_1\frac{4}{m}(3m-8)(8-m)t^{3m-9}) + \big(8b_1a(6-m) + b_2\frac{4}{m}(3m-8)(8-m)t^{3m-5}\Big\}dt + d\Big(\sum c_{\alpha,\beta}x^\alpha y^\beta\Big).$$

Looking at the above expression, we see there must exist a term in the summation $\sum c_{\alpha,\beta}x^\alpha y^\beta$ such that the value of $dx^\alpha y^\beta$ will cancel the term of order $2m-1$ in ω'. So for this term, $\beta = 2$ and $\alpha = 0$.

Since

$$dy^2 = \Big\{2mt^{2m-1} + 2(3m-8)t^{3m-9} + \left(\frac{3m-8}{m}(4m-8) + (2m-8)\right)t^{4m-17} + \cdots\Big\}dt,$$

we have

$$\nu\left((b_1x + b_2x^2)\omega - \frac{2b_1(8-m)}{m}dy^2\right) > \nu((b_1x + b_2x^2)\omega).$$

So

$$\omega' = \left\{ \left(8b_1 a(6-m) + b_2\frac{4}{m}(3m-8)(8-m)\right)t^{3m-5} - \frac{2b_1(8-m)}{m}\left(\frac{3m-8}{m}(4m-8) + (2m-8)\right)t^{4m-17} + \cdots \right\} dt + dh$$

for some $h \in \mathcal{O}$.

If $m > 12$, then $3m-5 < 4m-17$. Hence $\nu(\omega') = 3m-5 = \nu(y\omega)$ since $a \neq 0$. So the proof is complete in this case.

If $m = 9$ or $m = 11$, then $4m-17 < 3m-5$. Hence $\nu(\omega')+1 = 4(m-4) \in S$, and the proof is complete in this case too. □

We have proved the following theorem.

Theorem 17. *Let C be an irreducible algebroid plane curve. If $g = 1$ and $r(C) = 2$, then there exist two coprime positive integers n and m with $n < m$ such that C is analytically equivalent to a curve with a parametrization either of the form,*

$$x = t^n, \quad y = t^m + t^{(n-1)m-3n},$$

or of the form,

$$x = t^4, \quad y = t^m + t^{2m-8} + \frac{3m-8}{2m}t^{3m-16} + at^{3m-12},$$

where $m > 8$ and $a \in K$, or else of the form,

$$x = t^n, \quad y = t^m + t^{(n-2)m-2n} + at^{(n-1)m-3n},$$

where $n \geq 5$ and $m > 2n/(n-3)$ and where a is any element in K.

Conversely, any curve so parametrized has $r(C) = 2$ and $g = 1$.

Corollary 18. *Let C be an irreducible algebroid plane curve. If $g = 1$ and $r(C) = 2$, then there exist coprime integers m and n greater than 2 such that either C is analytically equivalent to the curve,*

$$X^n - Y^m + X^{n-2}Y^{m-3},$$

or C is analytically equivalent to some member of the following family of curves:

$$X^n - Y^m + X^{n-3}Y^{m-2} + \sum_{k=2}^{2+[m/n]} a_k X^{n-2}Y^{m-k}$$

where, in this case, $n \geq 4$and $m > 2n/(n-3)$ and $a_k \in K$.

Proof. This assertion follows from the theorem above and from [**5**, 1.5]. □

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